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A Fermat principle for stationary space–times and applications to light rays ☆

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Abstract

We present an extension of the classical Fermat principle in optics to stationary space–times. This principle is applied to study the light rays joining an event with a timelike curve. Existence and multiplicity results of light rays are proved. Moreover, Morse Relations relating the set of rays to the topology of the space–time are obtained, by using the number of conjugate points of the ray. The results hold also for stationary space–times with boundary, in particular the Kerr space–time outside the stationary limit surface.

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1. Introduction and statement of the results

In General Relativity a gravitational field is described by a 4-dimensional Lorentzian manifold (space–time) $(M, \langle\cdot,\cdot\rangle)$, where M is a smooth connected manifold and $\langle\cdot,\cdot\rangle$ is a Lorentzian metric, i.e. a metric tensor having index 1 (cf., e.g., Ref. [19]). The points of a space–time are usually called events.

In the study of the geometrical and physical properties of a space–time, geodesic curves play a basic role. We recall that a smooth curve on a Lorentzian manifold γ: $]a, b[\to M$ is said to be a *geodesic*, if

$$D_s\dot{\gamma}=0\,, \qquad (1.1)$$

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where $\dot{\gamma}$ is the tangent vector field along γ, and $D_s\dot{\gamma}$ is the covariant derivative of $\dot{\gamma}$ along γ, with respect to the Lorentzian metric $\langle \cdot, \cdot \rangle$.

It is well known that if γ is a geodesic, there exists a real number $E(\gamma)$ such that for any $s \in]a, b[$:

$$E(\gamma) = \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle .$$

The geodesic γ is called timelike, lightlike or spacelike if $E(\gamma)$ is negative, null or positive, respectively.

The most interesting cases, from a physical point of view, occur when $E(\gamma)$ is nonpositive (such geodesics are called causal). Timelike geodesics represent, in a space–time, the world lines of free falling particles (i.e. only gravity acts), while lightlike geodesics represent the trajectories of light rays. (A spacelike geodesic has no physical meaning, because travelling on it, a particle should be faster than light. However, they are useful in the study of the geometry of a Lorentzian manifold, for instance in the geodesic connectedness.)

Global properties about geodesics have been widely investigated. Avez and Seifert were the first to show that they every couple of causally related points of a globally hyperbolic Lorentzian manifold are joined by a causal geodesic (see Refs. [1,24]). Multiplicity results for timelike geodesics joining two causally related points on certain globally hyperbolic Lorentzian manifolds have been proved in Ref. [25]. Moreover, a Morse theory for the timelike geodesics joining two causally related points of a globally hyperbolic manifold has been developed in Ref. [25] on the space of the causal piecewise C^1-path.

Recently, some results about the geodesic connectedness of Lorentzian manifolds have been obtained by global variational methods (see, for instance, Refs. [5,6,14]).

Moreover, in Ref. [25] Uhlenbeck has developed (on the space of the null piecewise C^1-path) a Morse theory for the lightlike geodesics joining an event (the source of light), with a timelike curve (the observer), on a globally hyperbolic manifold. Other existence results for lightlike geodesics have recently been obtained in Ref. [9] on static space–times.

In the papers [1,24,25], the global hyperbolicity of the Lorentzian manifold is basic. However, many physically interesting Lorentzian manifolds are not globally hyperbolic, because they have a topological boundary (for instance, they are open subsets of a larger manifold).

Some results on the existence and multiplicity of timelike geodesics, and the geodesic connectedness for static and stationary Lorentzian manifolds with boundary, have recently been obtained in Refs. [3,4,12]. These results can be applied to physically relevant space–times as the Schwarzschild, Reissner–Nordström and Kerr space–times.

In this paper we prove existence and multiplicity results of lightlike geodesics joining an event with a timelike curve of Lorentzian product manifolds with boundary. Moreover, we develop a Morse theory for such geodesics. We consider a Lorentzian product manifold $(\tilde{M}, \langle\cdot,\cdot\rangle_z)$, such that

$$\tilde{M} = \tilde{M}_0 \times \mathbb{R}\,, \tag{1.2}$$

where $\tilde{M}_0$ is a smooth manifold, and $\langle\cdot,\cdot\rangle$ is conformal to a stationary metric, i.e. $\langle\cdot,\cdot\rangle_z$ is defined as follows: *for any* $z=(x,t)\in\tilde{M}_0\times\mathbb{R}$ *and for any* $\zeta=(\xi,\tau)\in T_z\tilde{M}\equiv T_x\tilde{M}_0\times\mathbb{R}$,

$$\langle\zeta,\zeta\rangle_z = \alpha(x,t)\,[\langle\xi,\xi\rangle_x + 2\langle\delta(x),\xi\rangle_x\tau - \beta(x)\tau^2]\,, \tag{1.3}$$

where $\alpha(x,t)$ and $\beta(x)$ are smooth positive scalar fields on $\tilde{M}$ and $\tilde{M}_0$, respectively, $\langle\cdot,\cdot\rangle_x$ is a Riemannian metric on $\tilde{M}_0$ and $\delta(x)$ is a smooth vector field on $\tilde{M}_0$.

A classical example of a space–time satisfying (1.2), (1.3) is the Kerr space-time (cf. Ref. [15]), which is the solution of the Einstein equations in the empty space, corresponding to the gravitational field produced by an axially symmetric body rotating around its axis. Some open subsets of the Kerr space–time (and also of the Schwarzschild and Reissner–Nordström space–times, see Ref. [3]) have a topological boundary which is light-convex (see section 7), according to the following definition.

Definition 1.1. Let $(\tilde{M}, \langle\cdot,\cdot\rangle)$ be a Lorentzian manifold and M a connected open subset of $\tilde{M}$ with boundary ∂M. We say that M has a *light-convex boundary* ∂M, if any lightlike geodesic $z:[a,b]\to M\cup\partial M$ with $z(a), Z(b)\in M$, has support $z([a,b])\subset M$.

Remark 1.2. The notion of light-convex boundary is independent of conformal changes of the metric, because the lightlike geodesics are independent, up to a reparametrization, of a conformal change of the metric.

In this paper we consider a Lorentzian manifold $(\tilde{M}, \langle\cdot,\cdot\rangle)$ which satisfies (1.2), (1.3), and an open connected subset M_0 of $\tilde{M}_0$. Setting

$$M = M_0 \times \mathbb{R}\,,$$

M is an open connected subset of $\tilde{M}$, having boundary $\partial M = \partial M_0\times\mathbb{R}$. We shall assume that:

$$\partial M_0 \text{ is a smooth submanifold of } \tilde{M}_0\,; \tag{1.4}$$

$M_0\cup\partial M_0$ is complete with respect to the Riemannian structure of M_0 given by

$$\langle\cdot,\cdot\rangle_1 = \langle\cdot,\cdot\rangle_x/\beta(x) \tag{1.5}$$

(i.e. any geodesic $x:]a, b[\to M_0$ with respect to the Riemannian structure

$\langle\cdot,\cdot\rangle_1$, can be extended to a continuous curve $\bar{x}:[a,b]\to M_0\cup\partial M_0$).

Let $z_0=(x_0,t_0)$ be a point of $M=M_0\times\mathbb{R}$, $x_1\in M_0\setminus\{x_0\}$, and consider the (timelike) vertical line $\gamma=\{(x_1,s),s\in\mathbb{R}\}$. The first result of this paper is the following existence theorem.

Theorem 1.3. *Let M_0 satisfy* (1.4), (1.5). *Assume that:*

$$\partial M=\partial M_0\times\mathbb{R} \text{ is light-convex}\,; \tag{1.6}$$

$$\sup_{x\in M_0}\langle\delta(x),\delta(x)\rangle_x/\beta(x)<+\infty\,. \tag{1.7}$$

Then there exists a lightlike geodesic

$$z^+(s)=(x^+(s),t^+(s))\,:\,[0,1]\to M\,,$$

joining $z_0=(x_0,t_0)$ and the vertical line $\gamma=\{(x_1,s),s\in\mathbb{R}\}$ (i.e. $z^+(0)=z_0$ and $x^+(1)=x_1$), such that $t^+(1)>t_0$. Moreover, there exists another lightlike geodesic

$$z^-(s)=(x^-(s),t^-(s)):[0,1]\to M\,,$$

joining z_0 and γ, such that $t^-(1)<t_0$.

Remark 1.4. Notice that if $M_0\cup\partial M_0$ is complete with respect to the Riemannian structure $\langle\cdot,\cdot\rangle_x$ and $\sup_{x\in M_0}\langle\delta(x),\delta(x)\rangle_x<+\infty$, then (1.5) and (1.7) are certainly satisfied if

$$0<\nu\le\beta(x)\le N<+\infty \text{ for any } x\in M_0\,.$$

Then the open subsets of the Kerr space–time considered in Section 7 satisfy (1.4)–(1.7).

If the topology of M_0 is nontrivial, we have the following multiplicity result.

Theorem 1.5. *Assume that M_0 is non contractible and* (1.4)–(1.7) *hold. Then there exist two sequences $z_m^+=(x_m^+,t_m^+):[0,1]\to M$ and $z_m^-=(x_m^-,t_m^-)\,:\,[0,1]\to M$ of lightlike geodesics joining z_0 with $\gamma=\{(x_1,s),s\in\mathbb{R}\}$, such that*

$$\lim_{m\to\infty}t_m^+(1)=+\infty \quad\text{and}\quad \lim_{m\to\infty}t_m^-(1)=-\infty\,.$$

The conclusion of Theorem 1.5 suggests the following physical interpretation. Consider a star, whose spatial coordinate is x_0, which explodes at time t_0. An observer which is in x_1 (for example on the earth), observes the explosion in a sequence of "instants" $\{t_m\}_{m\in\mathbb{N}}$, $t_m\overset{m}{\to}+\infty$.

Remark 1.6. The results of Theorems 1.3 and 1.5 hold even if $x_0 = x_1$ when M_0 is compact, giving existence and multiplicity results of light rays $(x(s), t(s))$ whose spatial component $x(s)$ is a periodic curve.

The proofs of Theorems 1.3 and 1.5 rest upon a variational principle which is a Lorentzian version, adapted to stationary metrics, of the classical Fermat principle in optics (for other results on the Fermat principle in General Relativity we quote Ref. [22]). By means of this variational principle, the study of the lightlike geodesics joining $z_0 = (x_0, t_0)$ with $\gamma = \{(x_1, s), s \in \mathbb{R}\}$ is equivalent to the research of the critical points of a suitable functional defined on the infinite dimensional manifold $\Omega^1(x_0, x_1, M_0)$ consisting of the absolutely continuous curves on M_0, joining x_0 and x_1, and having square summable derivative. The critical values of such a functional are the time-intervals $t(1) - t_0$ of the lightlike geodesics $z(s) = (x(s), t(s))$. A similar variational principle has been used in Ref. [25] on the space of the piecewise C^1-path.

The variational principle in Theorem 2.8 allows one to get, under non-degeneracy conditions, a Morse theory for the lightlike geodesics joining $z_0 = (x_0, t_0)$ with the line $\gamma = \{(x_1, s), s \in \mathbb{R}\}$. This theory relates the set of such geodesics with the topology of the space of the continuous curves joining two given points of M.

Before stating the Morse relations, we recall the following definitions.

Definition 1.7. Let $(M, \langle \cdot, \cdot \rangle)$ be a Lorentzian manifold and z be a geodesic joining z_0 with z_1 in the interval $[0,1]$. A point $z(s)$, $s \in]0,1]$, is said to be *conjugate* to z_0 along z if there exists a vector field $\zeta \neq 0$ along $z|_{[0,s]}$ which is solution of the system

$$
\begin{aligned}
&D_s^2 \zeta + R(\dot{z}, \zeta)\dot{z} = 0, \\
&\zeta(0) = \zeta(s) = 0,
\end{aligned}
\tag{1.8}
$$

where $R(\cdot, \cdot)$ is the curvature tensor of $(M, \langle \cdot, \cdot \rangle)$. The maximal number of linearly independent solutions of (1.8) is called the *multiplicity* of $z(s)$.

Definition 1.8 Let $z : [0,1] \to M$ be a geodesic. The *geometric index* $\mu(z)$ is the number of conjugate points $z(s)$ ($s \in]0,1]$) to $z(0)$ along z, counted with their multiplicity.

Remark 1.9. The geometric index of a geodesic can be $+\infty$ (see Ref. [16]). However, it is always finite on a stationary Lorentzian manifold (sf. Theorems 6.1 and 6.2).

Remark 1.10. For any lightlike geodesic the notions of conjugate point, multiplicity of conjugate points and geometric index are independent of conformal changes of the metric (see also Remark 1.2).

We introduce now the notion of non-degeneracy for lightlike geodesics joining an event with a timelike curve.

Definition 1.11. A point $z_0=(x_0, t_0)$ and a timelike curve γ of a Lorentzian manifold satisfying (1.2), are said to be *nonconjugate* if every lightlike geodesic $z:[0,1]\to M$ joining z_0 with γ is nondegenerate, i.e. $z(0)$ and $z(1)$ are nonconjugate along z.

Now let X be a topological space and $\mathbb{F}$ a field. We denote by $P_r(X)\equiv P_r(X,\mathbb{F})$ the Poincaré polynomial

$$P_r(X)=\sum_{k=0}^{\infty}\beta_k(X,\mathbb{F})r^k\,, \tag{1.9}$$

where, for every integer k, $\beta_k(X,\mathbb{F})$ is the kth Betti number of X with coefficients in $\mathbb{F}$, i.e. $\beta_k(X,\mathbb{F})=\dim H_k(X,\mathbb{F})$, where $H_k(X,\mathbb{F})$ is the kth singular homology group of X with coefficients in $\mathbb{F}$. Since $\mathbb{F}$ is a field, $H_k(X,\mathbb{F})$ is a vector space.

The following Morse relations hold.

Theorem 1.12. *Assume that* (1.4)–(1.7) *hold. Assume also that* $z_0=(x_0, t_0)$ *and* $\gamma=\{(x_1,s),\ s\in\mathbb{R}\}$ *are nonconjugate, according to Definition* 1.11. *Let*

$$Z^+=\{z=(x,t)\ :\ [0,1]\to M,\ \textit{lightlike geodesic}$$
$$\textit{joining } z_0 \textit{ with } \gamma \textit{ and such that } t(1)>t_0\}\,.$$

Moreover let Ω be the space of the continuous curves joining x_0 and x_1 in M_0, equipped with the uniform topology. Then

$$\sum_{z\in Z^+} r^{\mu(z)}=P_r(\Omega)+(1+r)Q(r)\,, \tag{1.10}$$

where $Q(r)$ is a formal series with natural coefficients (possibly $+\infty$).

Theorem 1.12 holds also for the set Z^- of lightlike geodesics joining z_0 with γ in the past of z_0, i.e. such that $t(1)<t_0$.

If M_0 does not have a boundary, Theorems 1.3, 1.5 and 1.12 can be proved analogously by a simpler proof. The results proved in this paper were announced in Ref. [11].

2. The variational principle

In this section we prove the variational principle used in the proofs of Theorems 1.3, 1.5 and 1.12.

Let $(M, \langle\cdot,\cdot\rangle_z)$ be a Lorentzian manifold satisfying (1.2), (1.3). By the well known Nash embedding Theorem (cf. Ref. [18]), the Riemannian manifold (M_0, $\langle \cdot, \cdot \rangle_x$) is isometric to a submanifold of $\mathbb{R}^N$ (with N sufficiently large) equipped with the Euclidean metric. Hence, we can assume that M_0 is a submanifold of $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle_x$ is the Euclidean metric of $\mathbb{R}^N$, which will be denoted by $\langle \cdot, \cdot \rangle$.

Set $I = [0,1]$. For every $k \in \mathbb{N}$ let $H^{1,2}(I, \mathbb{R}^k)$ be the Sobolev space of the absolutely continuous curves, whose derivative is square summable. It is a Hilbert space with norm

$$\|x\|_1^2 = \|x\|^2 + \|\dot{x}\|^2 = \int_0^1 \langle x, x \rangle \, ds + \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds , \tag{2.1}$$

where $\dot{x}$ denotes the derivative of x and $\|\cdot\|$ the usual norm of $L^2(I, \mathbb{R}^N)$.

Now, let x_0 and x_1 be two points of M_0, and

$$\Omega^1 = \Omega^1(M_0, x_0, x_1) = \{x \in H^{1,2}(I, \mathbb{R}^N) \mid x(I) \subseteq M_0, x(0) = x_0, x(1) = x_1\} .$$

It is well known that Ω^1 is a submanifold of $H^{1,2}(I, \mathbb{R}^N)$ and, for every $x \in \Omega^1$, the tangent space at Ω^1 is

$$T_x\Omega^1 = \{\xi \in H^{1,2}(I, \mathbb{R}^N) \mid \xi(s) \in T_{x(s)} M_0 \text{ for any } s \in I, \xi(0) = \xi(1) = 0\}$$

(cf., e.g., Refs. [20,23]).

Now, let t_0, $t_1 \in \mathbb{R}$ and consider

$$H^{1,2}(t_0, t_1) = \{t \in H^{1,2}(I, \mathbb{R}) \mid t(0) = t_0, t(1) = t_1\} .$$

$H^{1,2}(t_0, t_1)$ is a closed affine submanifold of $H^{1,2}(I, \mathbb{R})$, whose tangent space is

$$H_0^{1,2}(I, \mathbb{R}) = \{\tau \in H^{1,2}(I, \mathbb{R}) \mid \tau(0) = \tau(1) = 0\} .$$

Finally, let $z_0 = (x_0, t_0)$ be a point of M, $x_1 \in M_0$ and $\lambda \in \mathbb{R}$. We consider the path space of the $H^{1,2}$-curves joining z_0 and (x_1, λ),

$$\mathscr{Z}_\lambda = \mathscr{Z}_\lambda(z_0, x_1) = \Omega^1 \times H^{1,2}(t_0, \lambda) .$$

Obviously, for every $z = (x, t) \in \mathscr{Z}_\lambda$, the tangent space to $\mathscr{Z}_\lambda$ is

$$T_z \mathscr{Z}_\lambda = T_x \Omega^1 \times H_0^{1,2}(I, \mathbb{R}) .$$

On the manifold $\mathscr{Z} = \Omega^1 \times H^{1,2}(I, \mathbb{R})$, consider the action integral $f \colon \mathscr{Z} \to \mathbb{R}$, given by

$$f(z) = \frac{1}{2} \int_0^1 \langle \dot{z}, \dot{z} \rangle_z \, ds ,$$

which is a a smooth functional on $\mathscr{Z}$. We put

$$f_\lambda = f|_{\mathscr{Z}_\lambda} .$$

It is well known that for every $\lambda \in \mathbb{R}$, the critical points of f_λ are the geodesics joining $z_0=(x_0, t_0)$ and $z_\lambda=(x_1, \lambda)$. The search of geodesics joining z_0 and z_λ, i.e. the critical points of f_λ, is more difficult than in the Riemannian case. Indeed, f_λ is strongly indefinite, and the Morse index of its critical points is $+\infty$.

In this section, we shall prove a variational principle which allows us to reduce the search of light rays joining $z_0=(x_0, t_0)$ with the timelike vertical line $\gamma=\{(x_1, s), s\in\mathbb{R}\}$, to the search of the critical points of a functional defined only on Ω^1, bounded from below, and whose critical values are the time intervals $t(1)-t_0$, of the light rays $(x(s), t(s))$. Without loss of generality, we can assume that $t_0=0$.

Since the lightlike geodesics and the notions of conjugate point, multiplicity of a conjugate point, and geometric index, are independent of conformal changes of the metric (cf. Remarks 1.2 and 1.10), we can divide the Lorentzian metric $\langle\cdot, \cdot\rangle_z$ by the factor $\alpha(x, t)\beta(x)$. So, we can consider the new metric (again denoted by $\langle\cdot, \cdot\rangle_z$):

$$\langle\zeta, \zeta\rangle_z=\langle\xi, \xi\rangle_1+2\langle\delta(x), \xi\rangle_1\tau-\tau^2 ,$$

for all $z=(x, t)\in M$ and $\zeta=(\xi, \tau)\in T_zM=T_xM_0\times\mathbb{R}$, where $\langle\cdot, \cdot\rangle_1$ denotes $\langle\cdot, \cdot\rangle_x/\beta(x)$.

Notice that, by (1.5), $M_0\cup\partial M_0$ is complete with respect to the Riemannian metric $\langle\cdot, \cdot\rangle_1$. Moreover, by the Nash Embedding Theorem, we can assume that

$$\langle\zeta, \zeta\rangle_z=\langle\xi, \xi\rangle+2\langle\delta(x), \xi\rangle\tau-\tau^2 , \tag{2.2}$$

where $\langle\cdot, \cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^N$.

Consider now the energy integral for the metric (2.2), i.e.

$$f(z)=\frac{1}{2}\int_0^1 \langle\dot z, \dot z\rangle_z\, ds=\frac{1}{2}\int_0^1 [\langle\dot x, \dot x\rangle+2\langle\delta(x), \dot x\rangle\dot t-\dot t^2]\, ds , \tag{2.3}$$

and, for any $\lambda\in\mathbb{R}$,

$$f_\lambda=f|_{\mathscr{Z}_\lambda} .$$

Let $(\partial f_\lambda/\partial x)(x, t) : T_x\Omega^1\to\mathbb{R}$ and $(\partial f_\lambda/\partial t)(x, t) : H_0^{1,2}(I, \mathbb{R})\to\mathbb{R}$ be the partial derivatives of f_λ. Moreover, let

$$N_\lambda=\{z=(x, t)\in\mathscr{Z}_\lambda \mid (\partial f_\lambda/\partial t)(x, t)=0\} .$$

The following lemma, essentially proved in Ref. [12], holds:

Lemma 2.1. *For any $\lambda\in\mathbb{R}$, N_λ is the graphic of the smooth map $\Phi_\lambda : \Omega^1\to H^{1,2}(0, \lambda)$, given by*

$$\Phi_\lambda(x)(s)=\int_0^s \langle\delta(x), \dot x\rangle\, dr+s\left(\lambda-\int_0^1 \langle\delta(x), \dot x\rangle\, dr\right). \tag{2.4}$$

(Recall that we have assumed that $t_0=0$.)

Now consider the restriction of f_λ to the graphic of Φ_λ, i.e. the functional $J_\lambda : \Omega^1 \to \mathbb{R}$,

$$J_\lambda(x) = f_\lambda(x, \Phi_\lambda(x)) , \tag{2.5}$$

the following variational principle, which is essentially contained in Theorem 2.2 of Ref. [12], holds.

Theorem 2.2. *Let $z = (x, t) \in \mathscr{Z}_\lambda$. Then the following propositions are equivalent:*
(*a*) $z = (x, t)$ *is a critical point of* f_λ;
(*b*) (*i*) $t = \Phi_\lambda(x)$,
(*ii*) x *is a critical point of* J_λ.
Moreover if (*a*) *or* (*b*) *is true,*

$$f_\lambda(z) = J_\lambda(x) . \tag{2.6}$$

By (2.3) and (2.4), the explicit formula for J_λ is

$$2J_\lambda(x) = \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds + \int_0^1 \langle \delta(x), \dot{x} \rangle^2 \, ds - \left(\lambda - \int_0^1 \langle \delta(x), \dot{x} \rangle \, ds \right)^2 . \tag{2.7}$$

Our goal is to find geodesics $z = (x, t)$ joining z_0 with a point (x_1, λ) and such that

$$f_\lambda(z) = J_\lambda(x) = 0 .$$

To this aim, we consider λ as a variable and the functional $H : \Omega^1 \times \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} H(x, \lambda) &= 2J_\lambda(x) \\ &= \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds + \int_0^1 \langle \delta(x), \dot{x} \rangle^2 \, ds - \left(\lambda - \int_0^1 \langle \delta(x), \dot{x} \rangle \, ds \right)^2 . \end{aligned} \tag{2.8}$$

Then, by Theorem 2.2., the search of the lightlike geodesics joining $z_0 = (x_0, 0)$ with the vertical line $\gamma = \{(x_1, s), s \in \mathbb{R}\}$ is equivalent to the search of the points $(x, \lambda) \in \Omega^1 \times \mathbb{R}$ satisfying

$$\begin{aligned} &H(x, \lambda) = 0 , \\ &\frac{\partial H}{\partial x}(x, \lambda) = 0 . \end{aligned} \tag{2.9}$$

The following theorem shows that the problem (2.9) is equivalent to the search of the critical points of a suitable functional $F : \Omega^1 \to \mathbb{R}$. The functional F can be defined using an abstract framework.

Theorem 2.3. *Let $\mathcal{M}$ be a Hilbert manifold and let $H : \mathcal{M}\times\mathbb{R}\to\mathbb{R}$ be a smooth functional. Put*

$$\mathcal{G}=\{(x,\lambda)\in\mathcal{M}\times\mathbb{R} \mid H(x,\lambda)=0\}, \tag{2.10}$$

and assume that $\mathcal{G}$ is the graphic of a smooth functional $F : \mathcal{M}\to\mathbb{R}$.
Then, if $\bar{x}$ is a critical point of F, then $(\bar{x}, F(\bar{x}))$ solves (2.9).

Proof. Since $\mathcal{G}$ is the graphic of the functional F, for every $x\in\mathcal{M}$, we have:

$$H(x,F(x))=0.$$

Differentiating gives

$$0=\frac{\partial H}{\partial x}(x,F(x))+\frac{\partial H}{\partial \lambda}(x,F(x))F'(x). \tag{2.11}$$

Then, if $\bar{x}$ is a critical point of F, by (2.11) $(\bar{x}, F(\bar{x}))$ solves (2.9). □

Remark 2.4. Observe that conversely, if $(\bar{x},\bar{\lambda})$ solves (2.9) and $(\partial H/\partial\lambda)(\bar{x},\bar{\lambda})\neq 0$, then $\bar{x}$ is a critical point of F, and $F(\bar{x})=\bar{\lambda}$.

Now, we apply the results of Theorem 2.3 and Remark 2.4 to the functional $H(x,\lambda)$ defined by (2.8). In this case, the equation $H(x,\lambda)=0$ is solved by

$$\lambda=\int_0^1 \langle\delta(x),\dot{x}\rangle\, ds\pm\sqrt{\int_0^1 \langle\dot{x},\dot{x}\rangle\, ds+\int_0^1 \langle\delta(x),\dot{x}\rangle^2\, ds}. \tag{2.12}$$

Then, the set $\mathcal{G}$ in Theorem 2.3 consists of two branches, $\mathcal{G}_+$ and $\mathcal{G}_-$, which are respectively the graphic of the functionals $F_+, F_- : \Omega^1\to\mathbb{R}$ defined as follows:

$$F_+(x)=\sqrt{\int_0^1 \langle\dot{x},\dot{x}\rangle\, ds+\int_0^1 \langle\delta(x),\dot{x}\rangle^2\, ds}+\int_0^1 \langle\delta(x),\dot{x}\rangle\, ds, \tag{2.13}$$

$$F_-(x)=-\sqrt{\int_0^1 \langle\dot{x},\dot{x}\rangle\, ds+\int_0^1 \langle\delta(x),\dot{x}\rangle^2\, ds}+\int_0^1 \langle\delta(x),\dot{x}\rangle\, ds. \tag{2.14}$$

Remark 2.5. Since $x_0\neq x_1$, the functionals F_+ and F_- are smooth. If $x_0=x_1$, the functional is not differentiable only at the constant curve x_0.

Remark 2.6. If H is defined by (2.8), for every $(x,\lambda)\in\mathcal{G}$, the assumption $(\partial H/$

$\partial\lambda)(x,\lambda)\neq 0$ of Remark 2.4 holds. Indeed if

$$\frac{\partial H}{\partial\lambda}(x,\lambda)=-2\left(\lambda-\int_0^1\langle\delta(x),\dot{x}\rangle\,ds\right)=0\,, \tag{2.15}$$

since $H(x,\lambda)=0$, (2.8) and (2.15) give

$$\int_0^1\langle\dot{x},\dot{x}\rangle\,ds+\int_0^1\langle\delta(x),\dot{x}\rangle^2\,ds=0\,,$$

which implies that x is the constant curve x_0.

Remark 2.7. By the Hölder inequality:

$$\left(\int_0^1\langle\delta(x),\dot{x}\rangle\,ds\right)^2\leq\int_0^1\langle\delta(x),\dot{x}\rangle^2\,ds\,. \tag{2.16}$$

Then, since $x_0\neq x_1$, (2.8), (2.13), (2.14) and (2.16) give

$$F_+(x)>0 \quad \text{for any } x\in\Omega^1\,, \tag{2.17}$$

$$F_-(x)<0 \quad \text{for any } x\in\Omega^1\,. \tag{2.18}$$

From Theorem 2.3 and Remarks 2.4–2.7, the following variational principle holds:

Theorem 2.8. *Let x be a critical point of F_+ and Φ_λ as in (2.4). Then, if $\lambda=F_+(x)$, $(x,\Phi_\lambda(x))$ is a lightlike geodesic joining $z_0=(x_0,0)$ with $(x_1,F_+(x))$. Moreover, if $z=(x,t)$ is a lightlike geodesic joining z_0 with $z_1=(x_1,t_1)$ and $t_1>0$, then x is a critical point of F_+ and $F_+(x)=t_1$.*

The same result holds for the critical points of F_- and the lightlike geodesics joining z_0 with $z_1=(x_1,t_1)$ and $t_1<0$.

Remark 2.9. Since, for every $x\in\Omega^1$,

$$\frac{\partial H}{\partial\lambda}(x,\lambda)=-2\left(\lambda-\int_0^1\langle\delta(x),\dot{x}\rangle\,ds\right),$$

then, if $x_0\neq x_1$, for every $x\in\mathcal{G}$ we have,

$$\frac{\partial H}{\partial\lambda}(x,\lambda)<0 \quad \text{for every } x\in\mathcal{G}_+\,, \tag{2.19}$$

$$\frac{\partial H}{\partial\lambda}(x,\lambda)>0 \quad \text{for every } x\in\mathcal{G}_-\,. \tag{2.20}$$

3. The penalization argument

In Section 2 we presented a variational principle for the lightlike geodesics joining $z_0=(x_0,0)$ and the timelike line $\gamma=\{(x_1,s), s\in\mathbb{R}\}$. For the search of such geodesics, it suffices to find the critical point of the functionals $F_\pm : \Omega^1\to\mathbb{R}$,

$$F_\pm(x)=\pm\sqrt{\int_0^1\langle\dot{x},\dot{x}\rangle\,ds+\int_0^1\langle\delta(x),\dot{x}\rangle^2\,ds}+\int_0^1\langle\delta(x),\dot{x}\rangle\,ds\,.$$

We shall consider only the functional F_+ because the functional F_- can be studied analogously. In this section we analyze some properties of F_+ that will be useful in proving Theorems 1.3, 1.5 and 1.12.

First of all, notice that, if $x_0\neq x_1$, F_+ is a smooth functional, while, if $x_0=x_1$, F_+ is not differentiable only at the constant curve $x(s)\equiv x_0$. Moreover the following Lemma holds:

Lemma 3.1. *Assume that*

$$D=\sup_{x\in M_0}\langle\delta(x),\delta(x)\rangle<+\infty\,. \tag{3.1}$$

Then

(*i*) $\inf_{x\in\Omega^1}F_+(x)>0$,

(*ii*) $\lim_{\|x\|_*\to+\infty}F_+(x)=+\infty$,

where, for every $x\in\Omega^1$, $\|x\|_*^2=\int_0^1\langle\dot{x},\dot{x}\rangle\,ds$.

Proof. By Remark 2.7, F_+ is a positive functional. Moreover

$$F_+(x)=\sqrt{\int_0^1\langle\dot{x},\dot{x}\rangle\,ds+\int_0^1\langle\delta(x),\dot{x}\rangle^2\,ds}+\int_0^1\langle\delta(x),\dot{x}\rangle\,ds$$

$$=\frac{\int_0^1\langle\dot{x},\dot{x}\rangle\,ds+\int_0^1\langle\delta(x),\dot{x}\rangle^2\,ds-\left(\int_0^1\langle\delta(x),\dot{x}\rangle\,ds\right)^2}{\sqrt{\int_0^1\langle\dot{x},\dot{x}\rangle\,ds+\int_0^1\langle\delta(x),\dot{x}\rangle^2\,ds}-\int_0^1\langle\delta(x),\dot{x}\rangle\,ds}$$

(by Remark 2.9)

$$\geq c_0\|\dot{x}\|_*^2/\|\dot{x}\|_*=c_0\|\dot{x}\|_*\,, \tag{3.2}$$

where c_0 is a suitable constant depending on D (cf. (3.1)).

From (3.2) we deduce (ii) and

$$F_+(x) \geq c_0 d(x_0, x_1) > 0 , \tag{3.3}$$

(where d is the Riemannian distance induced by the Riemannian metric $\langle \cdot, \cdot \rangle$ of M_0). Finally (3.3) gives (i). □

In order to find critical points of the functional F_+, the Palais–Smale compactness condition plays an important role.

We recall that a smooth functional $I : X \to \mathbb{R}$, defined on a Hilbert manifold X, is said to satisfy the Palais–Smale condition at the level $c \in \mathbb{R}$ ($(\text{P.S.})_c$) if every sequence $\{x_k\}_{k\in\mathbb{N}} \subset X$ such that

$$I(x_k) \underset{k}{\to} c , \tag{3.4}$$

$$\|I'(x_k)\| \underset{k}{\to} 0 , \tag{3.5}$$

possesses a converging subsequence. Here $\|\cdot\|$ denotes the norm induced on T_xX by the Riemannian metric on X and I' the gradient of f.

Since M_0 is not a complete Riemannian manifold (because of the presence of the boundary ∂M_0), $\Omega^1 = \Omega^1(M_0, x_0, x_1)$ (which is an open submanifold of $\tilde{\Omega}^1 = \Omega^1(\tilde{M}_0, x_0, x_1)$) is not complete. For this reason the functional F_+ does not satisfy the Palais–Smale condition. Indeed, a sequence in Ω^1 which satisfies (3.4) and (3.5), may converge to a curve x which "touches" the boundary of M_0, and therefore $x \notin \Omega^1$.

In order to overcome this difficulty, we introduce a penalization argument. Since ∂M_0 is a smooth submanifold of $\tilde{M}_0$, there exists a smooth function $\phi : \tilde{M}_0 \to \mathbb{R}$, such that

$$M_0 = \{x \in \tilde{M}_0 \mid \phi(x) > 0\} , \tag{3.6}$$

$$\partial M_0 = \{x \in \tilde{M}_0 \mid \phi(x) = 0\} , \tag{3.7}$$

$$\text{grad}\, \phi(x) \neq 0 \quad \text{for any } x \in \partial M_0 , \tag{3.8}$$

where grad $\phi(x)$ denotes the gradient of ϕ at x, with respect to the Riemannian structure $\langle \cdot, \cdot \rangle$. Moreover for any $z \in M$ we set

$$\Phi(z) \equiv \Phi(x, t) = \phi(x) .$$

Notice that, denoting by ∇ the gradient of Φ with respect to the Lorentz structure $\langle \cdot, \cdot \rangle_z$, we have

$$\nabla\phi(z) = (\text{grad}\, \phi(x), 0) .$$

For every $\varepsilon > 0$, consider the penalized functional $f_\varepsilon : \mathcal{Z} \to \mathbb{R}$,

$$f_\varepsilon(z) = f_\varepsilon(x, t) = f(z) + \varepsilon \int_0^1 \frac{1}{\phi^2(x)}\, \mathrm{d}s , \tag{3.9}$$

where f is defined by (2.1). Moreover, for any $\lambda \in \mathbb{R}$, let $f_{\varepsilon,\lambda}$ be the restriction of f_ε to $\mathscr{Z}_\lambda = \Omega^1 \times H^{1,2}(0, \lambda)$.

Since the penalization term does not depend on the variable t, the statement of Theorem 2.3 holds also for the functional $f_{\varepsilon,\lambda}$. Therefore the search of the critical points of $f_{\varepsilon,\lambda}$ is equivalent to the same problem for the functional

$$J_{\varepsilon,\lambda}(x) = J_\lambda(x) + \varepsilon \int_0^1 \frac{1}{\phi^2(x)} \, ds \,, \tag{3.10}$$

where J_λ is defined by (2.7). On the other hand, the functional $H_\varepsilon(x, \lambda) = J_{\varepsilon,\lambda}(x)$ satisfies the assumptions of Theorem 2.3.

Therefore considering the functional $F_\varepsilon : \Omega^1 \to \mathbb{R}$ given by

$$F_\varepsilon(x) = \sqrt{\int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds + \int_0^1 \langle \delta(x), \dot{x} \rangle^2 \, ds + \varepsilon \int_0^1 \frac{1}{\phi^2(x)} \, ds} + \int_0^1 \langle \delta(x), \dot{x} \rangle \, ds \,, \tag{3.11}$$

the statements of Theorems 2.3 and 2.8, Remarks 2.4–2.7 and 2.9 still hold for the functional F_ε.

In order to prove the Palais–Smale condition for the functional F_ε ($\varepsilon > 0$), the following lemmas are needed.

Lemma 3.2. *Let $(x_k)_{k \in \mathbb{N}}$ be a sequence of Ω^1, such that*
(i) there exist a sequence $(s_k)_{k \in \mathbb{N}}$ in $I = [0, 1]$, such that

$$\lim_{k \to +\infty} \phi(x_k(s_k)) = 0 \,;$$

(ii)

$$\sup_{k \in \mathbb{N}} \int_0^1 \langle \dot{x}_k, \dot{x}_k \rangle \, ds < +\infty \,.$$

Then

$$\lim_{k \to +\infty} \int_0^1 \frac{1}{\phi^2(x_k)} \, ds = +\infty \,. \tag{3.12}$$

For the proof see Ref. [4].

Lemma 3.3. *Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in Ω^1, which weakly converges in $H^{1,2}(I,\mathbb{R}^N)$ to $x\in\Omega^1$. Then there exist two sequences $(\xi_k)_{k\in\mathbb{N}}$ and $(\nu_k)_{k\in\mathbb{N}}$ such that*

$$x_k - x = \xi_k + \nu_k\,, \tag{3.13}$$

$$\xi_k \in T_{x_k}\Omega^1, \qquad \nu_k \in H_0^{1,2}(I,\mathbb{R}^N)\,, \tag{3.14}$$

$$\xi_k \to 0 \quad \text{weakly in } H^{1,2}(I,\mathbb{R}^N)\,, \tag{3.15}$$

$$\nu_k \to 0 \quad \text{strongly in } H^{1,2}(I,\mathbb{R}^N)\,. \tag{3.16}$$

For the proof cf. Ref. [2].

Now, for any $a\in\mathbb{R}$ and for every $\varepsilon>0$ set

$$F_\varepsilon^a = \{x\in\Omega^1 \mid F_\varepsilon(x) \le a\}\,.$$

The following Proposition holds.

Proposition 3.4.
(i) *For every $\varepsilon>0$ and $a\in\mathbb{R}$, the set F_ε^a is a complete metric subspace of Ω^1.*
(ii) *For every $\varepsilon>0$ and $c\in\mathbb{R}^+$, F_ε satisfies* (P.S.)$_c$.

Proof. Let us prove (i). Let $(x_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in F_ε^a. Then $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence also in $H^{1,2}(I,\mathbb{R}^N)$, hence it converges to $x\in H^{1,2}(I,\mathbb{R}^N)$. (In particular (x_k) converges uniformly to x.) Moreover $(x_k)_{k\in\mathbb{N}}$ satisfies (ii) of Lemma 3.2. Therefore Lemma 3.2 gives

$$\inf\{\phi(x_k(s)),\, k\in\mathbb{N},\, s\in I\} > 0\,,$$

hence $x\in\Omega^1$ and the proof of (i) is complete.

Let us prove (ii). Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in Ω^1 such that

$$F_\varepsilon(x_k) \underset{k}{\longrightarrow} c\,, \tag{3.17}$$

$$\|F_\varepsilon'(x_k)\| \underset{k}{\longrightarrow} 0\,. \tag{3.18}$$

By (i) of Lemma 3.1, we can assume $c>0$. By (3.17) and (ii) of Lemma 3.1, $\{x_k\}_{k\in\mathbb{N}}$ is bounded in $H^{1,2}(I,\mathbb{R}^N)$; hence, up to considering a subsequence, it weakly converges to $x\in H^{1,2}(I,\mathbb{R}^N)$. Moreover, by (3.17) and Lemma 3.2, $x\in\Omega^1$.

In order to prove that $\{x_k\}_{k\in\mathbb{N}}$ converges to x strongly, consider $\{\xi_k\}_{k\in\mathbb{N}}$ and $\{\nu_k\}_{k\in\mathbb{N}}$ as in Lemma 3.3. From (3.18) and the definition of F_ε it follows that

$$
\begin{aligned}
\mathrm{o}(1)\|\xi_k\| &= F'_\varepsilon(x_k)[\xi_k] \\
&= \frac{1}{\sqrt{\int_0^1 \langle \dot{x}_k, \dot{x}_k\rangle\, \mathrm{d}s + \int_0^1 \langle \delta(x_k), \dot{x}_k\rangle^2\, \mathrm{d}s + \varepsilon \int_0^1 \frac{1}{\phi^2(x_k)}\, \mathrm{d}s}} \\
&\times \Bigg[\int_0^1 \langle \dot{x}_k, \dot{\xi}_k\rangle\, \mathrm{d}s + \int_0^1 \langle \delta(x_k), \dot{x}_k\rangle [\langle \delta'(x_k)\xi_k, \dot{x}_k\rangle + \langle \delta(x_k), \dot{\xi}_k\rangle]\, \mathrm{d}s \\
&- \int_0^1 \frac{\varepsilon}{\phi^3(x_k)} \langle \operatorname{grad}\phi(x_k), \xi_k\rangle\, \mathrm{d}s \Bigg] \\
&+ \int_0^1 \langle \delta'(x_k)\xi_k, \dot{x}_k\rangle\, \mathrm{d}s + \int_0^1 \langle \delta(x_k), \dot{\xi}_k\rangle\, \mathrm{d}s .
\end{aligned}
$$

Since $c > 0$, $\{x_k\}_{k\in\mathbb{N}}$ weakly converges to x and $(\xi_k)_{k\in\mathbb{N}}$ weakly converges to 0 in $H^{1,2}(I, \mathbb{R}^N)$ (and therefore the two convergencies are also uniform), we have

$$
\mathrm{o}(1) = \int_0^1 \langle \dot{x}_k, \dot{\xi}_k\rangle\, \mathrm{d}s + \int_0^1 \langle \delta(x_k), \dot{x}_k\rangle \langle \delta(x_k), \dot{\xi}_k\rangle\, \mathrm{d}s ,
$$

and from (3.13)

$$
\begin{aligned}
\mathrm{o}(1) &= \int_0^1 \langle \dot{x} + \dot{\nu}_k, \dot{\xi}_k\rangle\, \mathrm{d}s + \int_0^1 \langle \dot{\xi}_k, \dot{\xi}_k\rangle\, \mathrm{d}s \\
&\quad + \int_0^1 \langle \delta(x_k), \dot{x} + \dot{\nu}_k\rangle \langle \delta(x_k), \dot{\xi}_k\rangle\, \mathrm{d}s + \int_0^1 \langle \delta(x_k), \dot{\xi}_k\rangle^2\, \mathrm{d}s . \qquad (3.19)
\end{aligned}
$$

Then, by the weak convergence of x_k to x and of ξ_k to 0, and by the strong convergence of ν_k to 0 in $H^{1,2}(I, \mathbb{R}^n)$, (3.19) implies

$$
\mathrm{o}(1) = \int_0^1 \langle \dot{\xi}_k, \dot{\xi}_k\rangle\, \mathrm{d}s + \int_0^1 \langle \delta(x_k), \dot{\xi}_k\rangle^2\, \mathrm{d}s ,
$$

and, in particular,

$$
\int_0^1 \langle \dot{\xi}_k, \dot{\xi}_k\rangle\, \mathrm{d}s \underset{k}{\longrightarrow} 0 .
$$

Hence $\{\xi_k\}_{k\in\mathbb{N}}$ strongly converges to 0 in $H^{1,2}(I, \mathbb{R}^N)$, so $\{x_k\}_{k\in\mathbb{N}}$ strongly converges to x. □

4. Proof of Theorem 1.3

In this section we shall prove Theorem 1.3. Towards this goal, we need some estimates on the critical points of the penalized functionals F_ε, $\varepsilon>0$.

For every $\varepsilon>0$, let x_ε be a critical point of F_ε, such that

$$F_\varepsilon(x_\varepsilon)\le M\,, \tag{4.1}$$

where M is a constant independent of ε. By Theorem 2.8, x_ε is a critical point of $J_{\varepsilon,\lambda_\varepsilon}$ (cf. (3.10)), where $\lambda_\varepsilon=F_\varepsilon(x_\varepsilon)$ and $J_{\varepsilon,\lambda_\varepsilon}(x_\varepsilon)=0$.

Let $t_\varepsilon=\Phi_{\lambda_\varepsilon}(x_\varepsilon)$, (cf. (2.4) for the definition of Φ_λ, $\lambda\in\mathbb{R}$), then the curve $z_\varepsilon=(x_\varepsilon, t_\varepsilon)$ is a critical point of $f_{\varepsilon,\lambda_\varepsilon}=f_\varepsilon|_{\mathscr{Z}_{\lambda_\varepsilon}}$ (cf. Theorem 2.2) and $f_{\varepsilon,\lambda_\varepsilon}(z_\varepsilon)=0$.

Hence, for every $\zeta=(\xi,\tau)\in T_{z_\varepsilon}\mathscr{Z}_{\lambda_\varepsilon}\equiv T_{x_\varepsilon}\Omega^1\times H_0^{1,2}(I,\mathbb{R})$,

$$0=f'_{\varepsilon,\lambda_\varepsilon}(z_\varepsilon)[\zeta]=f'_{\lambda_\varepsilon}(z_\varepsilon)[\zeta]-\varepsilon\int_0^1 \frac{\langle \mathrm{grad}\,\phi(x_\varepsilon),\xi\rangle}{\phi^3(x_\varepsilon)}\,\mathrm{d}s \tag{4.2}$$

As proved in Ref. [12], z_ε is a smooth curve and satisfies the equations

$$-\nabla_s\dot z_\varepsilon=\frac{2\varepsilon}{\phi^3(x_\varepsilon)}\nabla\Phi(z_\varepsilon)\quad\text{for any } s\in I\,, \tag{4.3}$$

where $\Phi(x,t)=\phi(x)$, and $\nabla\Phi$ is the gradient of Φ with respect to the Lorentzian metric $\langle\cdot,\cdot\rangle_z$. Multiplying both terms of (4.3) by $\dot z$, gives the existence of a constant H_ε such that

$$H_\varepsilon=\tfrac12\langle\dot z_\varepsilon(s),\dot z_\varepsilon(s)\rangle_z-\frac{\varepsilon}{\phi^2(x_\varepsilon(s))}\quad\text{for any } s\in I\,. \tag{4.4}$$

Integrating (4.4) in the interval I gives

$$H_\varepsilon=\frac12\int_0^1\langle\dot z_\varepsilon,\dot z_\varepsilon\rangle-\varepsilon\int_0^1\frac{1}{\phi^2(x_\varepsilon)}\,\mathrm{d}s=f_{\varepsilon,\lambda_\varepsilon}(z_\varepsilon)-2\varepsilon\int_0^1\frac{1}{\phi^2(x_\varepsilon)}\,\mathrm{d}s$$

$$=-2\varepsilon\int_0^1\frac{1}{\phi^2(x_\varepsilon)}\,\mathrm{d}s<0\,. \tag{4.5}$$

The following estimates on the family $(z_\varepsilon)_{\varepsilon>0}$ hold.

Lemma 4.1. *For every $\varepsilon\in\,]0,1]$, let x_ε be a critical point of F_ε, such that (4.1) holds. Moreover let $\lambda_\varepsilon=F(x_\varepsilon)$, $t_\varepsilon=\Phi_{\lambda_\varepsilon}(x_\varepsilon)$ and $z_\varepsilon=(x_\varepsilon,t_\varepsilon)$. Then*

$$(i) \quad \sup_{\varepsilon \in\,]0,1]} \|x_\varepsilon\|_1 < +\infty \quad (cf.\ (2.1))\ ,$$

$$(ii) \quad \sup_{\varepsilon \in\,]0,1]} \|t_\varepsilon\|_1 < +\infty\ .$$

Proof. (i) follows from (4.1) and (3.2). Moreover by (2.4) with $t_0=0$,

$$\dot{t}_\varepsilon(s) = \langle \delta(x_\varepsilon), \dot{x}_\varepsilon \rangle + \left(\lambda_\varepsilon - \int_0^1 \langle \delta(x_\varepsilon), \dot{x}_\varepsilon \rangle \, \mathrm{d}s\right). \tag{4.6}$$

Then (4.6) and (i) imply a uniform bound of $(t_\varepsilon)_{\varepsilon \in]0,1]}$ in $L^2(I, \mathbb{R})$, proving (ii). □

Now consider the multiplier in Eq. (4.3),

$$\mu_\varepsilon(s) = \frac{2\varepsilon}{\phi^3(x_\varepsilon(s))}\ .$$

The following lemma holds:

Lemma 4.2. *Assume* (4.1). *Then*

$$\sup_{\varepsilon \in]0,1]} \|\mu_\varepsilon\|_{L^\infty} < +\infty\ . \tag{4.7}$$

Proof. For every $\varepsilon \in]0,1]$, let

$$h_\varepsilon(s) = \Phi(z_\varepsilon(s)) = \phi(x_\varepsilon(s))$$

and s_ε be a minimum point for h_ε in I. Clearly we have just to prove (4.7) whenever

$$\inf_{\varepsilon \in]0,1]} \phi(x_\varepsilon(s_\varepsilon)) = 0\ . \tag{4.8}$$

By Lemma 4.1. the families $(x_\varepsilon)_{\varepsilon \in]0,1]}$ and $(t_\varepsilon)_{\varepsilon \in]0,1]}$ have supports that are uniformly bounded in $\mathbb{R}^N$ and $\mathbb{R}$, respectively. Hence, there exists a constant $c_1 > 0$ such that

$$H_\Phi(z_\varepsilon(s_\varepsilon))[\dot{z}_\varepsilon, \dot{z}_\varepsilon] \le c_1 [\langle \dot{x}_\varepsilon(s_\varepsilon), \dot{x}(s_\varepsilon) \rangle + \dot{t}_\varepsilon^2(s_\varepsilon)]\ , \tag{4.9}$$

where $H_\Phi(z) : T_zM \times T_zM \to \mathbb{R}$ denotes the Lorentzian Hessian of the function Φ (see, e.g., Ref. [19] for the definition). Moreover, since 0 is a regular value of ϕ, there exists another constant $c_2 > 0$ (independent of ε), such that, if ε is small enough:

$$\langle \nabla\Phi(z_\varepsilon(s_\varepsilon)),\nabla\Phi(z_\varepsilon(s_\varepsilon))\rangle_z=\langle \operatorname{grad}\phi(x_\varepsilon(s_\varepsilon)),\operatorname{grad}\phi(x_\varepsilon(s_\varepsilon))\rangle\geq c_2>0\,. \tag{4.10}$$

Since s_ε is a minimum point of h_ε, by (4.9) and (4.3),

$$\begin{aligned}0\leq h''(s_\varepsilon)&=H_\Phi(z_\varepsilon(s_\varepsilon))[\dot z_\varepsilon(s_\varepsilon),\dot z_\varepsilon(s_\varepsilon)]+\langle \nabla\Phi(z_\varepsilon(s_\varepsilon)),D_s\dot z_\varepsilon(s_\varepsilon)\rangle_z\\&\leq c_1[\langle \dot x_\varepsilon(s_\varepsilon),\dot x(s_\varepsilon)\rangle+\dot t_\varepsilon(s_\varepsilon)^2]\\&\quad-\frac{2\varepsilon}{\phi^3(x_\varepsilon(s_\varepsilon))}\langle \operatorname{grad}\phi(x_\varepsilon(s_\varepsilon)),\operatorname{grad}\phi(x_\varepsilon(s_\varepsilon))\rangle\,.\end{aligned}$$

Hence, by (4.10), if ε is small enough,

$$\mu_\varepsilon(s_\varepsilon)=\frac{2\varepsilon}{\phi^3(x_\varepsilon(s_\varepsilon))}\leq\frac{c_1}{c_2}[\langle \dot x_\varepsilon(s_\varepsilon),\dot x_\varepsilon(s_\varepsilon)\rangle+\dot t_\varepsilon(s_\varepsilon)^2]\,. \tag{4.11}$$

On the other hand, by (4.6), (1.7) and (4.1) (recall that $\lambda_\varepsilon=F_\varepsilon(x_\varepsilon)$),

$$\dot t_\varepsilon(s_\varepsilon)^2\leq c_3+c_4\langle \dot x(s_\varepsilon),\dot x_\varepsilon(s_\varepsilon)\rangle\,, \tag{4.12}$$

where c_3 and c_4 are positive constants independent of ε. Now, by (4.4) and (2.1),

$$\begin{aligned}&\langle \dot x_\varepsilon(s_\varepsilon),\dot x_\varepsilon(s_\varepsilon)\rangle\\&\quad=2H_\varepsilon-2\langle\delta(x_\varepsilon(s_\varepsilon)),\dot x_\varepsilon(s_\varepsilon)\rangle\dot t_\varepsilon(s_\varepsilon)+\dot t_\varepsilon(s_\varepsilon)^2+\frac{\varepsilon}{\phi^2(x_\varepsilon(s_\varepsilon))}\,,\end{aligned} \tag{4.13}$$

and substituting (4.6) in (4.13), (3.1), (4.1), (4.5) and Lemma 4.1 give

$$\langle \dot x_\varepsilon(s_\varepsilon),\dot x_\varepsilon(s_\varepsilon)\rangle\leq c_5+c_6\frac{\varepsilon}{\phi^2(x_\varepsilon(s_\varepsilon))}\,, \tag{4.14}$$

where c_5 and c_6 are positive constants independent of ε.

Finally (4.11), (4.12) and (4.14) give

$$\mu_\varepsilon(s_\varepsilon)=\frac{2\varepsilon}{\phi^3(x_\varepsilon(s_\varepsilon))}\leq c_7+c_8\frac{\varepsilon}{\phi^2(x_\varepsilon(s_\varepsilon))}\,,$$

where c_7 and c_8 are positive constants independent of ε, giving (4.7). □

Corollary 4.3.

$$\lim_{\varepsilon\to0}\left\|\frac{\varepsilon}{\phi^2(x_\varepsilon(s))}\right\|_{L^\infty}=0\,.$$

Corollary 4.4. *The family of real functions $\mu_\varepsilon(s)=2\varepsilon/\phi^3(x_\varepsilon(s))$ weakly converges to $\mu(s)$ in $L^2(I,\mathbb{R})$. Moreover $\mu(s)$ is positive almost everywhere and if*

$$\inf\{\phi(x_\varepsilon(s_0)\ :\ \varepsilon\in\,]0,1]\}>0\,,$$

then $\mu(s)\equiv 0$ *in a neighborhood of* s_0.

From Lemmas 4.1 and 4.2, it is possible to deduce the following:

Proposition 4.5. *Let* x_ε *be a critical point of* F_ε *satisfying (4.1). Choose* $\lambda_\varepsilon = F_\varepsilon(x_\varepsilon)$ *and* $t_\varepsilon = \Phi_{\lambda_\varepsilon}(x_\varepsilon)$ *. Then there exists a sequence* $\varepsilon_k \xrightarrow[k]{} 0^+$*, such that, setting* $x_k = x_{\varepsilon_k}$ *and* $t_k = t_{\varepsilon_k}$,

(i) $\{x_k\}_{k\in\mathbb{N}}$ *converges to* x *in* $H^{1,2}(I,\mathbb{R}^N)$,

(ii) $\{t_k\}_{k\in\mathbb{N}}$ *converges to* t *in* $H^{1,2}(I,\mathbb{R})$,

(iii) $x(s)\in\Omega^1(M_0\cup\partial M_0, x_0, x_1)\subset\tilde{\Omega}^1=\Omega^1(\tilde{M}_0, x_0, x_1)$,

(iv) *if* $z=(x,t)$, *then for every* $\zeta=(\xi,\tau)\in T_x\tilde{\Omega}^1\times H_0^{1,2}(I,\mathbb{R})$,

$$\int_0^1 \langle \dot{z},\dot{\zeta}\rangle_z\,\mathrm{d}s = \int_0^1 \mu(s)\langle \nabla\Phi(z),\zeta\rangle_z\,\mathrm{d}s\,, \tag{4.15}$$

(v) $x\in H^{2,2}(I,\mathbb{R}^N)$ *and* $t\in H^{2,2}(I,\mathbb{R}^N)$, *consequently* x *and* t *are of class* C^1.

Proof. By Lemma 4.1 there exist two sequences $x_k\equiv x_{\varepsilon_k}$ and $t_k\equiv t_{\varepsilon_k}$, weakly convergent to x and t, respectively. Arguing as in the proof of Proposition 3.4 allows us to get the strong convergence, proving (i) and (ii). Moreover, (iii) follows by the assumption that $M_0\cup\partial M_0$ is complete with respect to the Riemannian metric $\langle\cdot,\cdot\rangle$. Now let us prove (iv).

Let $\zeta=(\xi,\tau)\in T_x\tilde{\Omega}^1\times H_0^{1,2}(I,\mathbb{R})$. Moreover, let ξ_k be the orthogonal projection of ξ on the tangent space of M_0 at x_k, so $\xi_k\in T_{x_k}\tilde{\Omega}^1=T_{x_k}\Omega^1$ (because the support of x_k is included in M_0). Then

$$\begin{aligned} 0 &= f'_{k,\lambda_k}(z_k)[(\xi_k,\tau)] \\ &= \int_0^1 [\langle \dot{x}_k,\dot{\xi}_k\rangle + \langle \delta'(x_k)\xi_k,\dot{x}_k\rangle\dot{t}_k + \langle\delta(x_k),\dot{\xi}_k\rangle\dot{t}_k \\ &\quad + \langle\delta(x_k),\dot{x}_k\rangle\dot{\tau}_k - \dot{t}_k\dot{\tau}_k]\,\mathrm{d}s \\ &\quad - \int_0^1 \mu_k(s)\langle \operatorname{grad}\phi(x_k),\xi_k\rangle\,\mathrm{d}s\,. \end{aligned} \tag{4.16}$$

Since ξ_k weakly converges to ξ in $H^{1,2}(I,\mathbb{R}^N)$ (cf., e.g., Ref [2]), taking the limit in (4.16) gives (4.15).

Finally by (4.15) and the regularity argument used in Lemma 4.7 of Ref. [12], we deduce that $x\in H^{2,2}(I,\mathbb{R}^N)$ and $t\in H^{2,2}(I,\mathbb{R})$, proving (v). □

Lemma 4.6. *Assume that M has light-convex boundary. Let x_ε be a critical point of F_ε, such that (4.1.) holds. With the same notation of Proposition* 4.5, *there exists a sequence $\{z_k=(x_k,t_k)\}_{k\in\mathbb{N}}$ converging in $H^{1,2}$ to a lightlike geodesic $z=(x,t)$ in M such that*

$$x(0)=x_0,\quad x(1)=x_1,\quad t(0)=0,\quad t(1)=F_+(x).$$

In particular, by Theorem 2.8, *x is a critical point of F_+.*

Proof. Consider a sequence $\{z_k\}_{k\in\mathbb{N}}$ given by Proposition 4.5 and its limit curve $z=(x,t)$. Using the light-convexity of ∂M we shall show that the limit curve $z=(x,t)$ does not touch the boundary ∂M and it is a lightlike geodesic, proving Lemma 4.6.

Towards this goal let us begin by noting that, by (4.15),

$$\langle \nabla_s\dot z,\dot z\rangle_z=-\mu(s)\langle \nabla\Phi(z),\dot z\rangle_z \quad \text{for almost every } s\in I. \tag{4.17}$$

Fix $s\in I$. From Corollary 4.4,

$$z(s)\notin\partial M \quad \text{implies} \quad \langle \nabla_s\dot z(s),\dot z(s)\rangle_{z(s)}=0.$$

Suppose $z(s)\in\partial M$. Then s is a minimum point of $\Phi(z(s))$ and $s\in\,]0,1[$. Then, since z is of class C^1 (cf. (v) of Proposition 4.5) and $s\in\,]0,1[$, $\langle\nabla\Phi(z),\dot z\rangle=0$. Therefore

$$\langle \nabla_s\dot z(s),\dot z(s)\rangle_{z(s)}=0 \quad \textit{for almost every } s\in I.$$

Hence, since z is of class C^1, there exists a constant $E\in\mathbb{R}$ such that

$$\langle \dot z(s),\dot z(s)\rangle_{z(s)}=E \quad \text{for any } s\in I. \tag{4.18}$$

Since for every $k\in\mathbb{N}$,

$$0=f_{\varepsilon_k,\lambda_k}(z_k)=\int_0^1\langle\dot z_k,\dot z_k\rangle_z\,ds+\varepsilon_k\int_0^1\frac{1}{\phi^2(x_k)}\,ds,$$

Corollary 4.3 and (i), (ii) of Proposition 4.5 give

$$\int_0^1\langle\dot z,\dot z\rangle_z\,ds=0.$$

Therefore $E=0$ and z is a lightlike curve.

Now, let s_0 be a point such that $z(s_0)\in\partial M$ and $\mu(s_0)$ is well defined (recall that μ is defined almost everywhere). Since $s_0\in\,]0,1[$ is a minimum point of $h(s)=\Phi(z(s))$ we have,

$$0\le h''(s_0)=H_\Phi(z(s_0))[\dot z(s_0),\dot z(s_0)]+\langle\nabla\Phi(z(s_0)),\nabla_s\dot z(s_0)\rangle_z$$

(by (4.15))

$$=H_\Phi(z(s_0))[\dot{z}(s_0),\dot{z}(s_0)]-\mu(s_0)\langle \text{grad}\,\phi(x(s_0)),\text{grad}\,\phi(x(s_0))\rangle\;.$$

Then

$$\mu(s_0)\langle \text{grad}\,\phi(x(s_0)),\text{grad}\,\phi(x(s_0))\rangle \le H_\Phi(z(s_0))[\dot{z}(s_0),\dot{z}(s_0)]\;. \quad (4.19)$$

Since $\dot{z}(s_0)\in T_{z(s_0)}\partial M$ is a lightlike vector and ∂M is light convex (cf. Definition 1.1) it is easy to see that

$$H_\Phi(z(s_0))[\dot{z}(s_0),\dot{z}(s_0)]\le 0\;;$$

therefore, by (4.19),

$$\mu(s_0)\langle \text{grad}\,\phi(x(s_0)),\text{grad}\,\phi(x(s_0))\rangle \le 0\,,$$

hence

$$\mu(s_0)\le 0\,.$$

On the other hand, by Corollary 4.4 $\mu(s)\ge 0$ almost everywhere, then

$$\mu(s)=0 \quad \text{for almost every } s\in I\,.$$

Then z satisfies the equation

$$\nabla_s\dot{z}=0 \quad \text{for almost every } s\in I\,.$$

which implies (using standard regularity arguments) that z is smooth and satisfies

$$\nabla_s\dot{z}=0 \quad \text{for any } s\in I\,.$$

Then z is a lightlike geodesic. Moreover, z cannot touch the boundary, because ∂M is light convex (cf. Definition 1.1) and $x_0, x_1\notin\partial M_0$. □

Proof of Theorem 1.3. For every $\varepsilon\in\,]0,1]$ the functional F_ε is bounded from below and satisfies the Palais–Smale condition at every level $c\in\mathbb{R}^+$. Then F_ε has a minimum point x_ε, such that

$$0<\inf_{\Omega^1} F\le \min_{\Omega^1} F_\varepsilon = F_\varepsilon(x_\varepsilon)\le F_\varepsilon(\bar{x})\le F_1(\bar{x})\,,$$

where $\bar{x}$ is a fixed C^1-curve joining x_0 with x_1 in M_0. Applying Lemma 4.6 gives immediately the proof of Theorem 1.3. □

5. Proof of Theorem 1.5

In this section we shall prove Theorem 1.5. By the variational principle proved in Section 2, it will be sufficient to prove the existence of a sequence $\{x_k\}_{k\in\mathbb{N}}$ of critical points of F_+ such that

$$\lim_{k \to +\infty} F_+(x_k) = +\infty . \tag{5.1}$$

Proof of Theorem 1.5. Assuming that M_0 is not contractible, Fadell and Husseini have proved in Ref. [10] that there exists a sequence $\{K_m\}_{m \in \mathbb{N}}$ of compact subsets of Ω^1, such that

$$\lim_{m \to +\infty} \text{cat}_{\Omega^1} (K_m) = +\infty , \tag{5.2}$$

where $\text{cat}_{\Omega^1}(K_m)$ is the Lusternik and Schnirelman category of K_m in Ω^1, i.e., the minimal integer positive number of closed contractible (in Ω^1) subsets of Ω^1 which cover Ω^1 (cf., e.g., Ref. [23] for definition and standard properties of the Lusternik and Schnirelmann category).

The same proof of Lemma 4.3 of Ref. [4] shows that

$$\text{cat}_{\Omega^1} (F_+^a) < +\infty \quad \text{for every } a \in \mathbb{R} , \tag{5.3}$$

where $F_+^a = \{x \in \Omega^1 \mid F_+(x) \leq a\}$. Now fix $\alpha \in \mathbb{R}$, and set

$$F_{\varepsilon,\alpha} = \{x \in \Omega^1 \mid F_\varepsilon(x) \geq \alpha\} .$$

For any $B \subset F_\varepsilon^\alpha = \{x \in \Omega^1 : F_\varepsilon(x) \leq \alpha\}$, since $F_\varepsilon^\alpha \subset F_+^\alpha$, by the monotonicity of the Lusternik and Schnirelmann category,

$$\text{cat}_{\Omega^1}(B) \leq \text{cat}_{\Omega^1}(F_+^\alpha) .$$

Therefore choosing

$$m = \text{cat}_{\Omega^1}(F_+^\alpha) + 1$$

gives

$$B \cap F_{\varepsilon,\alpha} \neq \emptyset \quad \text{for any } B \cap \Omega^1 \quad \text{such that } \text{cat}_{\Omega^1}(B) \geq m . \tag{5.4}$$

By (5.2), since there exists K_m, compact subset of Ω^1, such that

$$\text{cat}_{\Omega^1}(K) \geq m ,$$

the number

$$c_{m,\varepsilon} = \inf\{\sup F_\varepsilon(B) : \text{cat}_{\Omega^1}(B) \geq m\} \tag{5.5}$$

is well defined and

$$c_{m,\varepsilon} \leq \sup F_\varepsilon(K_m) \leq F_1(K_m) ,$$

while, by (5.4),

$$c_{m,\varepsilon} \geq \alpha .$$

Moreover, by well known critical point theorems (cf., e.g., Refs. [21,22]), since

F_ε satisfies the Palais–Smale condition at the level $c_{m,\varepsilon}$, there exists a critical point x_ε of F_ε, such that $F_\varepsilon(x_\varepsilon)=c_{m,\varepsilon}$. Then, by Lemma 4.6, there exists a subsequence of the family $(x_\varepsilon)_{\varepsilon\in]0,1]}$ converging to a critical point x_α of F_+. Since

$$F_+(x_\alpha)\geq\alpha\ ,$$

by the arbitrarity of α (5.1) is proved and the proof of Theorem 1.5 is complete. □

6. Proof of Theorem 1.12

In this section we shall prove Theorem 1.12 and get the Morse relations for the lightlike geodesics joining $(x_0, 0)$ and the timelike curve $\gamma=\{(x_1, s), s\in\mathbb{R}\}$, assuming that $(x_0, 0)$ and γ are nonconjugate (cf. Definition 1.9).

We begin recalling some results proved in Ref. [13] for stationary Lorentzian manifolds, which extend the analogous results proved in Ref. [8] for static Lorentzian manifolds (cf. also Ref. [17]).

Let $z=(x, t)$: $[0,1]\to M$ be a geodesic on a stationary Lorentzian manifold. Then x is a critical point of the functional J_λ, where $\lambda=t(1)$. For the geodesic z the geometric index $\mu(z,f)$, i.e. the number of conjugate points to $z(0)$ along $z|_{]0,1[}$, counted with their multiplicity (see Definition 1.9), is well defined. In the same way the geometric index $\mu(x, J_\lambda)$ for a critical point x of J_λ can be defined (cf. Ref. [8,13]). The following Theorem is essentially proved in Ref. [8]:

Theorem 6.1. *Let $z=(x, t)$ be a geodesic joining two points $z_0=(x, t_0)$ and $z_1=(x_1, t_1)$. Then*

$$\mu(z,f)=\mu(x, J_\lambda)\ , \tag{6.1}$$

where $\lambda=t(1)$.

Moreover, the Morse index Theorem holds for the functional J_λ defined on Ω^1. We recall that the Morse index of a critical point x of a smooth functional J (denoted by $m(x,J)$) defined on a Hilbert manifold X, is the maximal dimension of the subspaces of T_xX on which the Hessian $J''(x)$: $T_xX\times T_xX\to\mathbb{R}$ is negative definite.

The following result is a generalization of the Morse index Theorem for the geodesics of a Riemannian manifold and its proof is in Ref. [13].

Theorem 6.2. *Let x be a critical point of J_λ, then the Morse index $m(x, J_\lambda)$ is finite, and*

$$\mu(x, J_\lambda)=m(x, J_\lambda)\ . \tag{6.2}$$

Notice that Theorems 6.1 and 6.2 imply that the geometric index of a geodesic joining two given events of a stationary Lorentzian manifold is finite (see Remark 1.9).

To prove Theorem 1.12 the following preliminary results are needed.

Lemma 6.3. *let $\bar{x}$ be a critical point of the functional F_+, $\lambda = F_+(\bar{x})$, $m(\bar{x}, F_+)$ and $m(\bar{x}, J_\lambda)$ be the Morse indices of $\bar{x}$ as critical point of F_+ and J_λ, respectively. Then*

$$m(\bar{x}, F_+) = m(\bar{x}, J_\lambda) . \tag{6.3}$$

Proof. Differentiating the equality $H(x, F_+(x)) \equiv 0$ (cf. (2.8), (2.9) and Theorem 2.3) gives

$$\frac{\partial H}{\partial x}(x, F_+(x)) + \frac{\partial H}{\partial \lambda}(x, F_+(x)) F'_+(x) = 0 \quad \text{for any } x \in \Omega^1 . \tag{6.4}$$

Differentiating (6.4) gives

$$\begin{aligned} 0 = &\frac{\partial^2 H}{\partial x^2}(x, F_+(x)) + \frac{2\partial^2 H}{\partial \lambda \partial x}(x, F_+(x))[F'_+(x)] \\ &+ \frac{\partial^2 H}{\partial \lambda^2}(x, F_+(x))[F'_+(x)][F'_+(x)] + \frac{\partial H}{\partial \lambda}(x, F_+(x)) F''_+(x) ; \end{aligned}$$

therefore, since $F'_+(\bar{x}) = 0$,

$$0 = \frac{\partial^2 H}{\partial x^2}(\bar{x}, F_+(\bar{x})) + \frac{\partial H}{\partial \lambda}(\bar{x}, F_+(\bar{x})) F''_+(\bar{x})$$

(by (2.8) and (2.9))

$$= J''_\lambda(\bar{x}) + \frac{\partial H}{\partial \lambda}(\bar{x}, F_+(\bar{x})) F''_+(\bar{x}) .$$

Then

$$J''_\lambda(\bar{x}) = -\frac{\partial H}{\partial \lambda}(\bar{x}, F_+(\bar{x})) F''_+(\bar{x}) , \tag{6.5}$$

and combining (6.5) with (2.19) gives (6.3). □

Proposition 6.4. *If $(x_0, 0)$ and $\gamma = \{(x_0, s), s \in \mathbb{R}\}$ are nonconjugate, F_+ is a Morse function, i.e., its critical points are nondegenerate.*

Proof. Let x be a critical point of F_+. Then x is a critical point of J_λ, where $\lambda = F_+(x)$. By (6.5) and (2.19) x is a nondegenerate critical point for F_+ if and only if it is a nondegenerate critical point for J_λ. Now, let $z = (x, \phi_\lambda(x))$ be the lightlike geodesic associated to x. By the nondegeneracy assumptions on z_0 and $\gamma(s)$, z is a nondegenerate critical point of f_λ. Then, arguing as in Ref. [8] (where the relation between J''_λ and f''_λ is pointed out), we see that x is a nondegenerate critical point for J_λ, proving Proposition 6.4. □

Unfortunately, we cannot immediately get the Morse relations, because F_+ does not satisfy the Palais–Smale condition. For this reason the following penalized functional defined on Ω^1 is needed. Let

$$\psi(\sigma) = e^\sigma - (1+\sigma+\tfrac{1}{2}\sigma^2) ,$$

and, for every $\varepsilon > 0$,

$$\psi_\varepsilon(\sigma) = \begin{cases} \psi(\sigma - 1/\varepsilon) \text{ if } \sigma \geq 1/\varepsilon , \\ \quad 0 \qquad \text{if } \sigma < 1/\varepsilon . \end{cases}$$

Moreover, let $\tilde{F}_\varepsilon$ be the functional (3.11), where the penalization term $\varepsilon \int_0^1 (1/\phi^2(x))\,\mathrm{d}s$ is substituted by $\int_0^1 \psi_\varepsilon(1/\phi^2(x))\,\mathrm{d}s$, i.e.

$$\tilde{F}_\varepsilon(x) = \sqrt{\int_0^1 \langle \dot{x}, \dot{x} \rangle \,\mathrm{d}s + \int_0^1 \langle \delta(x), \dot{x} \rangle^2 \,\mathrm{d}s + \int_0^1 \psi_\varepsilon(1/\phi^2(x))\,\mathrm{d}s} + \int_0^1 \langle \delta(x), \dot{x} \rangle \,\mathrm{d}s . \tag{6.6}$$

Remark 6.5. Let $c \in \mathbb{R}$ be a regular value of F_+. If ε is sufficiently small, c is a regular value of $\tilde{F}_\varepsilon$. Indeed if $\{x_{\varepsilon k}\}_{k \in \mathbb{N}}$ is a sequence such that

$$\tilde{F}_{\varepsilon k}(x_{\varepsilon k}) = c \quad \text{and} \quad \tilde{F}'_{\varepsilon k}(x_{\varepsilon k}) = 0 \quad (\varepsilon_k \underset{k}{\longrightarrow} 0^+) ,$$

$\{x_{\varepsilon k}\}_{k \in \mathbb{N}}$ converges, up to a subsequence, to a critical point x of F_+ such that $F_+(x) = c$ (cf. Lemma 4.6).

Using the nondegeneracy of the critical points of F_+, the following lemma can be proved.

Lemma 6.6. *Assume that all the critical points of F_+ are nondegenerate and fix $c \in \mathbb{R}$. Then*

(i) on the sublevel $\{x \in \Omega^1 : F_+(x) \leq c\}$ there is only finite number of critical points of F_+;

(ii) there exists $\varepsilon_0 = \varepsilon_0(c) > 0$ such that for any $\varepsilon \in\,]0, \varepsilon_0]$, x is a critical point of F_+ on $\{x \in \Omega^1 : F_+(x) \leq c\}$, if and only if it is a critical point of $\tilde{F}_\varepsilon$ on $\{x \in \Omega^1 : \tilde{F}_\varepsilon(x) \leq c\}$. Moreover,

$$\tilde{F}_\varepsilon(x) = F_+(x) \quad \text{and} \quad m(x, \tilde{F}_\varepsilon) = m(x, F_+) . \tag{6.7}$$

Proof. Any critical point of F_+ is isolated (because it is nondegenerate). Moreover, the same proof of Proposition (3.4) (ii) and the light-convexity of ∂M_0 show that $\{x\in\Omega^1 : F_+(x)\le c \text{ and } F'_+(x)=0\}$ is compact, giving (i). The light-convexity of ∂M_0 and the form of the penalization term in $\tilde{F}_\varepsilon$ gives also (ii). □

From Lemma 6.6 the following Morse relations on the sublevels of $\tilde{F}_\varepsilon$ hold.

Theorem 6.7. *Assume that all critical points of F_+ are* nondegenerate. Let c be a regular value of F_+ *and set* $Z(F_+, c)=\{x\in F_+^c \mid F'_+(x)=0\}$. *Then there exists* $\varepsilon_0(c)>0$, *such that, for any* $\varepsilon\in\,]0,\varepsilon_0(c)]$,

$$\sum_{x\in Z(F_+,c)} r^{m(x,F_+)}=P_r(\tilde{F}_\varepsilon^c)+(1+r)Q_{\varepsilon,c}(r)\,, \tag{6.8}$$

where $P_r(\tilde{F}_\varepsilon^c)$ is the Poincaré polynomial of $\tilde{F}_\varepsilon^c=\{x\in\Omega^1 \mid \tilde{F}_\varepsilon(x)\le c\}$ and $Q_{\varepsilon,c}$ is a formal series with natural coefficients (possibly $+\infty$).

Proof. By Lemma 6.6, if ε is small enough, c is a regular value for $\tilde{F}_\varepsilon$ and $\tilde{F}_\varepsilon$ contains only nondegenerate critical points under the level c. Since $\tilde{F}_\varepsilon$ satisfies the Palais–Smale condition at every level, we have

$$\sum_{x\in Z(\tilde{F}_\varepsilon,c)} r^{m(x,\tilde{F}_\varepsilon)}=P_r(\tilde{F}_\varepsilon^c)+(1+r)Q_{\varepsilon,c}(r)\,,$$

for a suitable formal series $Q_{\varepsilon,c}$ (cf., e.g., Ref. [7]). Moreover, if ε is small enough, Lemma 6.5 gives also

$$Z(\tilde{F}_\varepsilon, c)=Z(F_+, c) \quad \text{and} \quad m(x,\tilde{F}_\varepsilon)=m(x,F_+) \quad \text{for any } x\in Z(F_+, c)\,,$$

proving (6.8). □

Now, since the critical points of F_+ do not touch the boundary, the form of the penalization term in (6.6) and the same techniques used in Ref. [13] allow us to get the following Lemmas.

Lemma 6.8. *For every regular value c of F_+ there exists $\varepsilon_0(c)>0$ such that, for any $\varepsilon\in\,]0,\varepsilon_0(c)]$ the sublevel F_+^c is a weak deformation retract of $\tilde{F}_\varepsilon^c$.*

Lemma 6.9. *Let $c_2>c_1$ be regular values for F_+. Then there exists $\varepsilon_0=\varepsilon_0(c_1, c_2)>0$ such that, for any $\varepsilon\in\,]0, \varepsilon_0]$ the pair $(F_+^{c_2}, F_+^{c_1})$ is a weak deformation retract of the pair $(\tilde{F}_\varepsilon^{c_2}, \tilde{F}_\varepsilon^{c_1})$.*

Now we are finally ready to prove Theorem 1.12.

Proof of Theorem 1.12. By (6.8) and Lemma 6.8, for any c, regular value of F_+,

$$\sum_{x\in Z(F_+,c)} r^{m(x,F_+)} = P_r(F^c) + (1+r)Q_c(r) , \tag{6.9}$$

where Q_c is a formal series with natural coefficients (possibly $+\infty$). Moreover, using standard arguments of algebraic topology, cf., e.g., Refs. [7,13], since F_+ is a Morse function and Lemma 6.9 holds, sending c to $+\infty$ gives

$$\sum_{x\in Z(F_+,+\infty)} r^{m(x,F_+)} = P_r(\Omega^1) + (1+r)Q(r) , \tag{6.10}$$

where again Q is a formal series with natural coefficients.

Now denote by $Z^+(z_0, \gamma)$ the set of the null geodesics $z=(x,t)$ (from $[0,1]$ to M) joining z_0 and γ and such that $t(1)>0$. Then by (6.10), Theorems 6.1, 6.2 and Lemma 6.3, we finally get

$$\sum_{z\in Z^+(z_0,\gamma)} r^{\mu(z)} = P_r(\Omega^1) + (1+r)Q(r) ,$$

proving Theorem 1.12. □

7. Applications to the Kerr space–time

Consider the Kerr space–time outside of the "stationary limit surface", i.e. the space–time

$$M=\{(r,\vartheta,\phi) : r>m+\sqrt{m^2-a^2\cos^2\vartheta}\}\times\mathbb{R} \tag{7.1}$$

with metric

$$ds^2=\lambda\left(\frac{dr^2}{\Delta}+d\vartheta^2\right)+(r^2+a^2)\sin^2\vartheta\, d\phi^2-dt^2+\frac{2mr}{\lambda}(a\sin^2\vartheta\, d\phi-dt)^2 , \tag{7.2}$$

where

$$\lambda=\lambda(\vartheta,r)\equiv r^2+a^2\cos^2\vartheta, \qquad \Delta=\Delta(r)\equiv r^2-2mr+a^2 .$$

Here $m>0$ represents the mass of the rotating body responsible for the gravitational field, ma is the angular momentum as measured from infinity (see, e.g., Ref [15]) and $a^2<m^2$. If $a=0$, the space–time (7.1) with metric (7.2) is the Schwarzschild space–time.

When a is small enough there are open subsets of M having light-convex boundary. More precisely the following proposition holds [whose proof can be carried out as in Ref. [17], using the Hessian of the function $\Phi_a(r,\theta)=\frac{1}{2}(r^2-2mr+a^2\cos^2\vartheta)$]:

Proposition 7.1. *Let r_0 be the smallest zero greater than $2m$ of the equation*

$$9m^2r^4(r-m)(r-2m)+(r-3m)[r^3+3m(r-m)(r-2m)]=0\,. \qquad (7.3)$$

Let $\varepsilon:\mathbb{R}^+\to\mathbb{R}^+$ *be a strictly decreasing function and*

$$\varepsilon_0=\lim_{a\to 0}\varepsilon(a)\,. \qquad (7.4)$$

Assume

$$0<\varepsilon_0<(r_0-m)^2-m^2\,. \qquad (7.5)$$

Then there exists $a_0>0$ *such that, for every* a *satisfying* $|a|\le a_0$, *the boundary of*

$$M_a=\{r>m+\sqrt{m^2+\varepsilon(a)-a^2\cos^2\vartheta}\}\times\mathbb{R} \qquad (7.6)$$

with metric (7.2), is light-convex.

The Kerr metric satisfies assumptions (1.5)–(1.7) for $|a|$ sufficiently small. Then Theorems 1.3, 1.5 and 1.12 hold for 7.1, if $|a|$ is sufficiently small.

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